

1. a)

Direct method:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & -4 & -6 & 1 & -1 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & -6 & 1 & -1 & 0 \\ 0 & 12 & 17 & -2 & 3 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -4 & -6 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 12 & 17 & -2 & 3 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow -R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & -6 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 12 & 17 & -2 & 3 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 12R_2} \left[\begin{array}{ccc|ccc} 1 & -4 & -6 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 41 & -2 & 3 & 12 \end{array} \right]$$

$$R_3 \rightarrow \frac{1}{41} R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -4 & -6 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{-2}{41} & \frac{3}{41} & \frac{12}{41} \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 + 4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -14 & 1 & -1 & -4 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{-2}{41} & \frac{3}{41} & \frac{12}{41} \end{array} \right]$$

$$\begin{aligned} R_1 &\rightarrow R_1 + 14R_3 \\ R_2 &\rightarrow R_2 + 2R_3 \end{aligned} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{41} & \frac{1}{41} & \frac{4}{41} \\ 0 & 1 & 0 & \frac{-4}{41} & \frac{6}{41} & \frac{-17}{41} \\ 0 & 0 & 1 & \frac{-2}{41} & \frac{3}{41} & \frac{12}{41} \end{array} \right] \Rightarrow C^{-1} = \frac{1}{41} \begin{bmatrix} 13 & 1 & 4 \\ -4 & 6 & -17 \\ -2 & 3 & 12 \end{bmatrix}$$

Adjoint method:

the matrix of cofactors:

$$\begin{aligned} &\begin{bmatrix} -\begin{vmatrix} 4 & 5 \\ -12 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ -12 & 0 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & 4 \end{vmatrix} \end{bmatrix} = \\ &= \begin{bmatrix} 13 & -4 & -2 \\ 1 & 6 & 3 \\ 4 & -17 & 12 \end{bmatrix} \quad \text{adj } C = \begin{bmatrix} 13 & -4 & -2 \\ 1 & 6 & 3 \\ 4 & -17 & 12 \end{bmatrix}^T = \begin{bmatrix} 13 & 1 & 4 \\ -4 & 6 & -17 \\ -2 & 3 & 12 \end{bmatrix} \end{aligned}$$

$$\det C = 3 \begin{vmatrix} 4 & 5 \\ -12 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} = 3 \cdot 13 + 2 = 41$$

$$\text{Therefore, } C^{-1} = \frac{1}{\det C} \cdot \text{adj } C = \frac{1}{41} \begin{bmatrix} 13 & 1 & 4 \\ -4 & 6 & -17 \\ -2 & 3 & 12 \end{bmatrix}.$$

b) A is 2×3 , C^{-1} is $3 \times 3 \Rightarrow AC^{-1}$ is 2×3

C^T is 3×3

We can not add AC^{-1} and C^T , since they have different dimensions.

c) $3A + B^T$ - not defined
 $2 \times 3 \quad 2 \times 2$ (different dimensions)

$$d) \quad AA^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + (-1)(-1) + 2 \cdot 2 & 1 \cdot 0 - 1 \cdot 3 + 2 \cdot 5 \\ 0 \cdot 1 + 3(-1) + 5 \cdot 2 & 0 \cdot 0 + 3 \cdot 3 + 5 \cdot 5 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 7 \\ 7 & 34 \end{bmatrix}.$$

$$2B^2 = 2 \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} = 2 \begin{bmatrix} 2 \cdot 2 + 3 \cdot 5 & 2 \cdot 3 + 3 \cdot 6 \\ 5 \cdot 2 + 6 \cdot 5 & 5 \cdot 3 + 6 \cdot 6 \end{bmatrix} =$$
$$= 2 \begin{bmatrix} 19 & 18 \\ 40 & 51 \end{bmatrix} = \begin{bmatrix} 38 & 36 \\ 80 & 102 \end{bmatrix}$$

$$AA^T - 2B^2 = \begin{bmatrix} -32 & -29 \\ -73 & -68 \end{bmatrix}$$

e) By a property of inverses, $(B^T)^{-1} = (B^{-1})^T$, therefore $(B^T)^{-1} - (B^{-1})^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which does not have an inverse, so the expression is not defined.

$$\textcircled{2} \text{ a) } \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 5R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & -9 & 14 \end{bmatrix}$$

$$\xrightarrow{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & -9 & 14 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 9R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow -\frac{1}{6}R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 9R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_8 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I \Rightarrow$$

$$A^{-1} = E_8 E_7 E_6 \dots E_1 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} \dots E_8^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 7 & 6 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 7R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -15 \end{bmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -15 \end{bmatrix} \xrightarrow{R_3 \rightarrow -\frac{1}{15}R_3} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 3R_3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 3R_3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = E_7 E_6 \dots E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} \dots E_7^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that in both a) and b) the elementary matrices are not unique (depends on what row operations you use).

3. a) First, find inverse of $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ 5 & 6 & 9 \end{bmatrix}$.

I will use the adjoint method.

(2nd row)

$$\det A = -2 \begin{vmatrix} 3 & -1 \\ 6 & 9 \end{vmatrix} - 4 \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} = -2 \cdot 33 - 4 \cdot (-9) = -30$$

$$A^{-1} = \frac{1}{-30} \begin{bmatrix} \begin{vmatrix} 0 & 4 \\ 6 & 9 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 6 & 9 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 4 \\ 5 & 9 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 5 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} -24 & -33 & 12 \\ 2 & 14 & -6 \\ 12 & 9 & -6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} -24 & -33 & 12 \\ 2 & 14 & -6 \\ 12 & 9 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} -279 \\ 102 \\ 87 \end{bmatrix} = \begin{bmatrix} \frac{279}{30} \\ -\frac{102}{30} \\ \frac{87}{30} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{93}{10} \\ -\frac{17}{5} \\ \frac{29}{10} \end{bmatrix}. \quad \Rightarrow \quad x = \frac{93}{10}, \quad y = -\frac{17}{5}, \quad z = \frac{29}{10}.$$

b) Find the inverse of $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(1st row)

$$\det A = 2 \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix} = 2 \cdot (-9) + (-12) = -30.$$

$$A^{-1} = -\frac{1}{30} \begin{bmatrix} \begin{vmatrix} 13 \\ 56 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 56 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 13 \end{vmatrix} \\ -\begin{vmatrix} 23 \\ 46 \end{vmatrix} & \begin{vmatrix} 12 & 0 \\ 46 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 45 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 45 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} -9 & 6 & -3 \\ -12 & 12 & -6 \\ -4 & -14 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} -9 & 6 & -3 \\ -12 & 12 & -6 \\ -4 & -14 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} -9a+6b-3c \\ -12a+12b-6c \\ -4a-14b+2c \end{bmatrix}$$

$$x = -\frac{1}{30}(-9a+6b-3c), \quad y = -\frac{1}{30}(-12a+12b-6c), \\ z = -\frac{1}{30}(-4a-14b+2c).$$

$$\textcircled{4.} \quad a) \left| \begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 18 \\ 7 & 0 & 2 & -18 \end{array} \right| \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 7R_1 \end{array} = \left| \begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -2 & -4 & 0 \\ 0 & -21 & -33 & -72 \end{array} \right| \begin{array}{l} R_2 \rightarrow -\frac{1}{2}R_2 \\ R_3 \rightarrow R_3 + 21R_2 \end{array} = -2 \left| \begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & -21 & -33 & -72 \end{array} \right| =$$

$$= -2 \left| \begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 9 & -72 \end{array} \right| = -2 \cdot 1 \cdot 1 \cdot 9 = -18.$$

$$b) \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 3 & 1 & 2 & 3 & 18 \\ 0 & 0 & 3 & 0 & -18 \\ -12 & 1 & 4 & 5 & -72 \end{array} \right| \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_4 \rightarrow R_4 + R_1 \end{array} = \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & -8 & -10 & -12 & -9 \\ 0 & 0 & 3 & 0 & -18 \\ 0 & 5 & 5 & 9 & -63 \end{array} \right| \begin{array}{l} R_2 \rightarrow -\frac{1}{2}R_2 \\ R_3 \rightarrow \frac{1}{3}R_3 \end{array} = -2 \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & 4 & 5 & 6 & 9 \\ 0 & 0 & 3 & 0 & -6 \\ 0 & 5 & 5 & 9 & -63 \end{array} \right| =$$

$$\begin{array}{l} R_4 \rightarrow R_4 - R_2 \\ R_2 \leftrightarrow R_4 \end{array} = -2 \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & 4 & 5 & 6 & 9 \\ 0 & 0 & 3 & 0 & -6 \\ 0 & 1 & 0 & 3 & -72 \end{array} \right| = -(-2) \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & 1 & 0 & 3 & -72 \\ 0 & 0 & 3 & 0 & -6 \\ 0 & 4 & 5 & 6 & 9 \end{array} \right| = 2 \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & 1 & 0 & 3 & -72 \\ 0 & 0 & 3 & 0 & -6 \\ 0 & 0 & 5 & -6 & 162 \end{array} \right| =$$

$$= 2 \cdot 3 \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & 1 & 0 & 3 & -72 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 5 & -6 & 162 \end{array} \right| \begin{array}{l} R_4 \rightarrow R_4 - 5R_3 \end{array} = 6 \left| \begin{array}{cccc|c} 1 & 3 & 4 & 5 & 9 \\ 0 & 1 & 0 & 3 & -72 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -6 & 162 \end{array} \right| = 6 \cdot 1 \cdot 1 \cdot 1 \cdot (-6) = -36.$$

$$\textcircled{5} \quad a) \quad \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 0 & 2 \end{vmatrix} \stackrel{(3\text{rd row})}{=} 7 \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 7 \cdot (3 \cdot 6 - 5 \cdot 4) + 2(1 \cdot 4 - 3 \cdot 2) = \\ = 7(-2) + 2(-2) = -18.$$

$$b) \quad \begin{vmatrix} 1 & 3 & 4 & 5 \\ 3 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 \\ -1 & 2 & 1 & 4 \end{vmatrix} \stackrel{(3\text{rd row})}{=} 3 \begin{vmatrix} 1 & 3 & 5 \\ 3 & 1 & 3 \\ -1 & 2 & 4 \end{vmatrix} \stackrel{(1\text{st row})}{=} 3 \left(1 \cdot \begin{vmatrix} 3 & 3 \\ -1 & 4 \end{vmatrix} - 3 \begin{vmatrix} 3 & 3 \\ -1 & 4 \end{vmatrix} + 5 \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} \right) \\ = 3(1 \cdot (-2) - 3 \cdot 15 + 5 \cdot 7) = 3 \cdot (-12) = -36.$$

$$\textcircled{6} \quad a) \quad \det A = -30 \quad (\text{see 3a})$$

$$|A_1| = \begin{vmatrix} 2 & 3 & -1 \\ 7 & 0 & 4 \\ 0 & 6 & 9 \end{vmatrix} \stackrel{(1\text{st column})}{=} 2 \begin{vmatrix} 0 & 4 \\ 6 & 9 \end{vmatrix} - 7 \begin{vmatrix} 3 & -1 \\ 6 & 9 \end{vmatrix} = 2(-24) - 7(33) = \\ = -279$$

$$|A_2| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ 5 & 0 & 9 \end{vmatrix} \stackrel{(2\text{nd col.})}{=} -2 \begin{vmatrix} 2 & 4 \\ 5 & 9 \end{vmatrix} + 7 \begin{vmatrix} 1 & -1 \\ 5 & 9 \end{vmatrix} = (-2)(-2) + 7 \cdot 4 = \\ = 102$$

$$|A_3| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 0 & 7 \\ 5 & 6 & 0 \end{vmatrix} \stackrel{(3\text{rd col.})}{=} 2 \begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} - 7 \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} = 2 \cdot 12 - 7 \cdot (-9) = 87.$$

$$x = \frac{|A_1|}{|A|} = \frac{-279}{-30} = \frac{93}{10}, \quad y = \frac{102}{-30} = -\frac{17}{5}, \quad z = \frac{87}{-30} = -\frac{29}{10}.$$

$$b) \quad \det A = -30$$

$$|A_1| = \begin{vmatrix} a & -1 & 0 \\ b & 1 & 3 \\ c & 5 & 6 \end{vmatrix} \stackrel{(1\text{st column})}{=} a \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} - b \begin{vmatrix} -1 & 0 \\ 5 & 6 \end{vmatrix} + c \begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} = -9a + 6b - 3c$$

$$|A_2| = \begin{vmatrix} 2 & a & 0 \\ 0 & b & 3 \\ 4 & c & 6 \end{vmatrix} \stackrel{(2\text{nd column})}{=} -a \begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix} + b \begin{vmatrix} 2 & 0 \\ 4 & 6 \end{vmatrix} - c \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = -12a + 12b - 6c$$

$$|A_3| = \begin{vmatrix} 2 & -1 & a \\ 0 & 1 & b \\ 4 & 5 & c \end{vmatrix} \stackrel{(3\text{rd col.})}{=} a \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} - b \begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} + c \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = -4a - 14b + 2c$$

$$x = -\frac{1}{30}(-9a + 6b - 3c), \quad y = -\frac{1}{30}(-12a + 12b - 6c), \quad z = -\frac{1}{30}(-4a - 14b + 2c)$$

7.

$$a) \begin{vmatrix} def \\ abc \\ gh i \end{vmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{3}R_3} \begin{vmatrix} def \\ abc \\ ghi \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{vmatrix} abc \\ def \\ ghi \end{vmatrix} = -3 \cdot 3 = -9$$

$$b) \begin{vmatrix} a & b & c \\ a+2d & b+2e & c+2f \\ g & h & i \end{vmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} 2 \begin{vmatrix} a & b & c \\ def \\ ghi \end{vmatrix} =$$

$$= 2 \cdot 3 = 6$$

$$c) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \det \left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T \right) = \det \left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = 3.$$

$$d) \begin{vmatrix} a & b & c \\ d & f & g \\ a-d & b-f & c-g \end{vmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_1} \begin{vmatrix} a & b & c \\ d & f & g \\ -d & -f & -g \end{vmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{vmatrix} a & b & c \\ d & f & g \\ 0 & 0 & 0 \end{vmatrix} = 0.$$